

# Microscopic Electron Models with Exact $SO(5)$ Symmetry

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We construct a class of microscopic electron models with exact  $SO(5)$  symmetry between antiferromagnetic and  $d$  wave superconducting ground states. There is an exact one-to-one correspondence between both single-particle and collective excitations in both phases.  $SO(5)$  symmetry breaking terms can be introduced and classified according to irreducible representations of the exact  $SO(5)$  algebra. The resulting phase diagram and collective modes are identical to that of the  $SO(5)$  nonlinear  $\sigma$  model.

PACS numbers: 74.20.Mn, 74.25.Ha, 71.10.-w

One of the most interesting features of the high  $T_c$  superconductivity is the close proximity and the interplay between the antiferromagnetic (AF) and the  $d$  wave superconducting (dSC) phases. Recently, a theoretical formalism was introduced based on the concept of a  $SO(5)$  symmetry between these two phases, and the resulting field-theoretical model describes the cuprate phase diagram and collective modes in a unified framework [1]. It was argued that the microscopic Hubbard model supports an approximate  $SO(5)$  symmetry [1–3].

In this note, we construct a class of microscopic electron models with exact  $SO(5)$  symmetry. In this model, degeneracy between the AF and dSC phases can be demonstrated exactly, and both the fermionic single-particle and the bosonic collective modes can be mapped onto each other with a precise one-to-one correspondence. This model can be used as a starting point around which  $SO(5)$  symmetry breaking interactions can be introduced and classified according to irreducible tensors of the  $SO(5)$  algebra. It is shown that the resulting phase diagram and the collective modes are similar to those obtained from the effective  $SO(5)$  nonlinear  $\sigma$  model with anisotropic couplings [1,4]. The purpose of this work is to demonstrate that the general  $SO(5)$  idea can be realized exactly by explicit microscopic Hamiltonians. The microscopic information extracted from this class of models, especially the behavior of the fermionic excitations across the AF/dSC transition would greatly complements the effective field theory approach. Within this class of models, we have a consistent microscopic theory of the AF/dSC transition. Since both the AF and the dSC states are stable infrared fixed points, it is plausible that one can deform the parameters so that the microscopic  $SO(5)$  models can also serve as a paradigm for a much more general class of AF/dSC transitions, including those occurring in the high  $T_c$  cuprates and 2D organics.

The first independent attempt to construct microscopic  $SO(5)$  invariant models was undertaken by Christopher Henley [5]. He independently made a crucial observation that if one replaces the standard  $d$  wave factor  $\cos p_x - \cos p_y$  by  $\text{sgn}(\cos p_x - \cos p_y)$ , the  $SO(5)$  algebra introduced in [1] closes exactly.

It is easy to write down many electron models with exact  $SU(2)$  spin rotation invariance, because the electron operator  $c_{\mathbf{p}\sigma}$  forms a natural spinor representation of the  $SU(2)$  algebra. In writing down  $SU(2)$  invariant models, all we have to do is to contract the spinor indices in a natural way. Therefore, the first step towards constructing a  $SO(5)$  invariant electron model is to find a natural definition of a  $SO(5)$  spinor. Spinor representations of the  $SO(5)$  Lie algebra can be easily constructed using the Clifford algebra of five  $4 \times 4$  Dirac matrices [6] satisfying  $\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}$  ( $a, b = 1, \dots, 5$ ), and the ten  $SO(5)$  rotation generators are given by  $\Gamma^{ab} = -i[\Gamma^a, \Gamma^b]$ . In this work we shall use the following explicit representation for the Clifford algebra:

$$\Gamma^1 = \begin{pmatrix} 0 & -i\sigma_y \\ i\sigma_y & 0 \end{pmatrix} \Gamma^{(2,3,4)} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & {}^t\vec{\sigma} \end{pmatrix} \Gamma^5 = \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}$$

where  $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  are the usual  $2 \times 2$  Pauli matrices, and  ${}^t$  means transposition. We define a 4 component spinor by

$${}^t\Psi_{\mathbf{p}} = \left\{ c_{\mathbf{p}\uparrow}, c_{\mathbf{p}\downarrow}, \phi_{\pi}(\mathbf{p})c_{-\mathbf{p}+\mathbf{Q},\uparrow}^\dagger, \phi_{\pi}(\mathbf{p})c_{-\mathbf{p}+\mathbf{Q},\downarrow}^\dagger \right\} \quad (1)$$

where  $\phi_{\pi}(\mathbf{p}) = \text{sgn}(\cos p_x - \cos p_y) = \pm 1$ , and  $\mathbf{Q} = (\pi, \pi)$ . Since we have two spin degrees of freedom at a given momentum  $\mathbf{p}$ , such a description must be redundant. Indeed, one can easily see that the spinors with momenta outside of the magnetic Brillouin zone is related to the spinors inside the magnetic Brillouin zone by an “R conjugation”

$$\Psi_{\mathbf{p}+\mathbf{Q}} = \phi_{\pi}(\mathbf{p})R\Psi_{-\mathbf{p}}^* \quad (2)$$

The  $R$  matrix is an invariant tensor of the  $SO(5)$  algebra enjoying the following properties:  $R\Gamma^aR = -{}^t\Gamma^a$ ,  $R\Gamma^{ab}R = {}^t\Gamma^{ab}$ . In our representation it takes the form  $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . (The existence of such a matrix is related to the fact that the spinor representation of the  $SO(5)$  Lie algebra is pseudo-real. The  $\sigma_y$  matrix plays a similar role for  $SO(3)$ ). The  $\Psi_{\mathbf{p}\alpha}$  spinors obey the anticommutation relations:

$$\{\Psi_{\mathbf{p}\alpha}^\dagger, \Psi_{\mathbf{p}'\beta}\} = \delta_{\alpha\beta}\delta_{\mathbf{p},\mathbf{p}'}, \quad (3)$$

$$\{\Psi_{\mathbf{p}\alpha}^\dagger, \Psi_{\mathbf{p}'\beta}^\dagger\} = \{\Psi_{\mathbf{p}\alpha}, \Psi_{\mathbf{p}'\beta}\} = -\phi_\pi(\mathbf{p})R_{\alpha\beta}\delta_{\mathbf{p}+\mathbf{p}',\mathbf{Q}}. \quad (4)$$

If we restrict  $\mathbf{p}$  and  $\mathbf{p}'$  to be both inside the magnetic Brillouin zone, the right hand side of the second equation vanishes and the  $\Psi_{\mathbf{p}\alpha}$  spinors commute in the same way as the  $c_{\mathbf{p}\sigma}$  spinors. They can be used to construct the  $SO(5)$  vector order parameter and the symmetry generators:  $n_a = \frac{1}{4}\sum_{\mathbf{p}} w_{\mathbf{p}} \Psi_{\mathbf{p}}^\dagger \Gamma^a \Psi_{\mathbf{p}+\mathbf{Q}}$ , and  $L_{ab} = \frac{1}{8}\sum_{\mathbf{p}} \Psi_{\mathbf{p}}^\dagger \Gamma^{ab} \Psi_{\mathbf{p}}$ . Here  $w_{\mathbf{p}} = w_{-\mathbf{p}}$ . Note that the definition of the  $\pi$  operators ( $L_{1(2,3,4)}, L_{(2,3,4)5}$ ) differs from the ones used in previous works [1–3], where they are electron pair operators on the nearest neighbor (n.n.) sites. The problem with this kind of definition is that the commutator algebra does not close, and generates longer-ranged bonds. Naively, the condition for the closure of the  $SO(5)$  algebra appears to be over-constrained. The present work and [5] start with electron pair operators with a long-ranged profile, given in real space by the lattice Fourier transform of  $\phi_\pi(\mathbf{p})$ ,

$$\phi_\pi(m, n) = \frac{2}{\pi^2} \frac{1 - (-)^{m+n}}{m^2 - n^2}, \quad (5)$$

where  $\mathbf{R} = (m, n)$  is a lattice point. It is truly remarkable that this simple choice closes the algebra exactly. Notice that while the  $\pi$  operators have long-ranged profiles, the dSC order parameter can still be short-ranged with suitable choices of  $w_{\mathbf{p}}$ . Under the  $SO(5)$  rotations generated by the  $L_{ab}$ ,  $\Psi_{\mathbf{p}}$  transforms as a proper  $SO(5)$  spinor

$$[L_{ab}, \Psi_{\mathbf{p}\alpha}] = -\frac{1}{4}(\Gamma^{ab})_{\alpha\beta} \Psi_{\mathbf{p}\beta} \quad (6)$$

for all values of  $\mathbf{p}$ . Using these spinors, exact  $SO(5)$  invariant Hamiltonians can be constructed simply by proper contraction of the spinor indices.

We start with the kinetic term, and write it as

$$H_{\text{kin}} = \sum_{\mathbf{p}, \sigma} \varepsilon_{\mathbf{p}} c_{\mathbf{p}, \sigma}^\dagger c_{\mathbf{p}, \sigma} = \frac{1}{2} \sum_{\mathbf{p}} \varepsilon_{\mathbf{p}} \Psi_{\mathbf{p}}^\dagger \Psi_{\mathbf{p}}. \quad (7)$$

We see that the property  $\varepsilon_{\mathbf{p}+\mathbf{Q}} = -\varepsilon_{\mathbf{p}}$ , valid for n.n. tight binding model, is crucial for this construction to work. In order to construct four-fermion interactions, we first note that a  $SO(5)$  spinor bilinear can in general be decomposed into a direct sum of a scalar, a vector, and an antisymmetric tensor, *i.e.*  $4 \times 4 = 1 + 5 + 10$ . Therefore, general  $SO(5)$  invariant four-fermion interactions can be expressed as

$$\begin{aligned} H_{\text{int}} = & \sum_{\mathbf{p}, \mathbf{p}', \mathbf{q}} V_1(\mathbf{p}, \mathbf{p}'; \mathbf{q}) (\Psi_{\mathbf{p}}^\dagger \Gamma^a \Psi_{\mathbf{p}+\mathbf{q}}) (\Psi_{\mathbf{p}'}^\dagger \Gamma^a \Psi_{\mathbf{p}'-\mathbf{q}}) \\ & + \sum_{\mathbf{p}, \mathbf{p}', \mathbf{q}} V_2(\mathbf{p}, \mathbf{p}'; \mathbf{q}) (\Psi_{\mathbf{p}}^\dagger \Gamma^{ab} \Psi_{\mathbf{p}+\mathbf{q}}) (\Psi_{\mathbf{p}'}^\dagger \Gamma^{ab} \Psi_{\mathbf{p}'-\mathbf{q}}) \\ & + \sum_{\mathbf{p}, \mathbf{p}', \mathbf{q}} V_0(\mathbf{p}, \mathbf{p}'; \mathbf{q}) (\Psi_{\mathbf{p}}^\dagger \Psi_{\mathbf{p}+\mathbf{q}}) (\Psi_{\mathbf{p}'}^\dagger \Psi_{\mathbf{p}'-\mathbf{q}}). \end{aligned} \quad (8)$$

Since  $L_{ab}(\mathbf{p}, \mathbf{q}) \equiv \Psi_{\mathbf{p}}^\dagger \Gamma^{ab} \Psi_{\mathbf{p}+\mathbf{q}}$ ,  $n_a(\mathbf{p}, \mathbf{q}) \equiv \Psi_{\mathbf{p}}^\dagger \Gamma^a \Psi_{\mathbf{p}+\mathbf{Q}+\mathbf{q}}$  and  $\rho(\mathbf{p}, \mathbf{q}) \equiv \Psi_{\mathbf{p}}^\dagger \Psi_{\mathbf{p}+\mathbf{q}}$  are the true  $SO(5)$  tensor, vector, and scalar, respectively, for any  $\mathbf{p}$  and  $\mathbf{q}$ , their inner products naturally gives a manifestly  $SO(5)$  invariant Hamiltonian.

Among three terms in  $H_{\text{int}}$ , we concentrate on the vector interaction (first term) in all subsequent analysis, and assume a factorizable form  $V_1(\mathbf{p}, \mathbf{p}'; \mathbf{q}) = -V_1(\mathbf{q})w_{\mathbf{p}}w_{\mathbf{p}'}$ ; This form is not necessary, but simplifies calculations. In real space,

$$\begin{aligned} H_{\text{int},1} = & -4 \sum_{\ell, n} V_1(\mathbf{R}_\ell - \mathbf{R}_n) e^{i\mathbf{Q} \cdot (\mathbf{R}_\ell - \mathbf{R}_n)} \\ & \times \left[ \vec{m}_\ell \cdot \vec{m}_n + \frac{1}{2} (\Delta_\ell^\dagger \Delta_n^\dagger + \Delta_\ell^\dagger \Delta_n) \right]. \end{aligned} \quad (9)$$

Here,  $\vec{m}_\ell$  and  $\Delta_\ell$  are Neel and  $d$ -wave pairing order parameters (operators) at site  $\ell \equiv \mathbf{R}_\ell$ , but with extended internal structures determined by  $w_{\mathbf{p}}$ . For the simplest choice  $w_{\mathbf{p}} = 1$ , they become

$$\vec{m}_\ell = \frac{1}{2} (\psi_\ell^\dagger \vec{\sigma} \psi_\ell - \chi_\ell^\dagger \vec{\sigma} \chi_\ell) e^{i\mathbf{Q} \cdot \mathbf{R}_\ell}, \quad (10)$$

$$\Delta_\ell^\dagger = \sum_j \phi_\pi(\mathbf{R}_\ell - \mathbf{R}_j) (c_{\ell\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{\ell\downarrow}^\dagger c_{j\uparrow}^\dagger). \quad (11)$$

Here we introduced two-component spinors  $\psi_\ell = {}^t(c_{\ell\uparrow}, c_{\ell\downarrow})$  and  $\chi_\ell = (-e^{i\mathbf{Q} \cdot \mathbf{R}_\ell}) \times {}^t(b_{\ell\uparrow}, b_{\ell\downarrow})$  with  $b_{\ell\sigma} = \sum_j \phi_\pi(\mathbf{R}_\ell - \mathbf{R}_j) c_{j\sigma}$ . The pair wave function for dSC condensate is described by  $\phi_\pi$  and is long-ranged. For the choice  $w_{\mathbf{p}} = |\cos p_x - \cos p_y|$ , we obtain

$$\vec{m}_\ell = \frac{e^{i\mathbf{Q} \cdot \mathbf{R}}}{2} \sum_i \phi_M(\mathbf{R}_\ell - \mathbf{R}_i) (\psi_i^\dagger \vec{\sigma} \psi_\ell - \chi_i^\dagger \vec{\sigma} \chi_\ell) \quad (12)$$

$$\Delta_\ell^\dagger = \sum_{i,j} \phi_M(\mathbf{R}_\ell - \mathbf{R}_i) \phi_\pi(\mathbf{R}_\ell - \mathbf{R}_j) (c_{i\uparrow}^\dagger c_{j\downarrow}^\dagger - c_{i\downarrow}^\dagger c_{j\uparrow}^\dagger) \quad (13)$$

where

$$\phi_M(m, n) = \frac{4}{\pi^2} \frac{1 + (-)^{m+n}}{[(m+n)^2 - 1][(m-n)^2 - 1]}. \quad (14)$$

The interaction between centers of mass of  $\vec{m}$  or  $\Delta$  fields is controlled by  $V_1(\mathbf{R})$ . If we take  $V_1(\mathbf{q})$  to be a  $\delta$  function at  $\mathbf{q} = \mathbf{Q}$ , the  $\Delta$ -part in  $H_{\text{int},1}$  becomes the usual BCS reduced Hamiltonian for n.n.  $d$ -wave pairing. If  $V_1(\mathbf{q})$  is taken to be a Lorentzian around  $\mathbf{q} = \mathbf{Q}$ , the real space form of the spin interaction resembles the potential induced by the AF paramagnon exchange [7–9].

It is not difficult to find degeneracy between AF and dSC states in the usual treatment of mean field theories. However, their excitation spectra are generally different, and quantum fluctuations may remove this degeneracy. In the  $SO(5)$  invariant models, symmetry not only ensures exact degeneracy of the ground states, but also ensures exact one-to-one correspondence between their excitation spectra. This fact is formulated as follows:

*Theorem 1:* If  $|\Psi_0\rangle$  is a ground state of a  $SO(5)$  invariant Hamiltonian with AF broken symmetry (say in the  $n_2$  direction), *i.e.*  $\langle \Psi_0 | n_a | \Psi_0 \rangle = \delta_{2,a} A$ , then  $|\Psi'_0\rangle = e^{i\frac{\pi}{2}L_{12}}|\Psi_0\rangle$  is a degenerate ground state with dSC broken symmetry (in  $n_1$  direction), *i.e.*  $\langle \Psi'_0 | n_a | \Psi'_0 \rangle = \delta_{1,a} A$ . Furthermore, all excited states of the AF ground state can be mapped to excited states of the dSC ground state at the same energy by the  $e^{i\frac{\pi}{2}L_{12}}$  operator.

The proof of this theorem is elementary, since  $L_{12}$  commutes with the Hamiltonian, and  $e^{-i\frac{\pi}{2}L_{12}}n_1e^{i\frac{\pi}{2}L_{12}} = n_2$ . In the following, we shall illustrate this powerful theorem in an explicit mean field calculation. We take a “generalized BCS reduced Hamiltonian” by selecting  $V_1(\mathbf{q}) = V_1\delta_{\mathbf{q},\mathbf{Q}}$  in the vector interaction. The Green’s function in the presence of a mean field  $\langle n_{\mathbf{p}}^a \rangle = \frac{1}{4}\langle \Psi_{\mathbf{p}}^\dagger \Gamma^a \Psi_{\mathbf{p}+\mathbf{Q}} \rangle$  is given by

$$G_{\alpha\beta}(\mathbf{p}, \mathbf{p}'; \omega) = -i \int dt e^{i\omega t} \langle T \Psi_{\mathbf{p},\alpha}(t) \Psi_{\mathbf{p}',\beta}^\dagger(0) \rangle \quad (15)$$

$$= \frac{(\omega + \varepsilon_{\mathbf{p}})\delta_{\alpha\beta}\delta_{\mathbf{p},\mathbf{p}'} + \Delta_{\mathbf{p}}^a \Gamma_{\alpha\beta}^a \delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2 - \varepsilon_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2 + i\delta}$$

where  $\Delta_{\mathbf{p}}^a = -16V_1w_{\mathbf{p}}\sum_{\mathbf{k}}w_{\mathbf{k}}\langle n_{\mathbf{k}}^a \rangle$ . This manifestly  $SO(5)$  invariant Green’s function explicitly shows that the AF quasi-particles can be mapped onto dSC quasi-particles. In particular, the AF Green’s function (in the  $n_2$  direction) can be obtained directly from the dSC Green’s function (in the  $n_1$  direction) by a simple rotation:  $G^{AF} = e^{-i\frac{\pi}{2}\Gamma_{12}}G^{SC}e^{i\frac{\pi}{2}\Gamma_{12}}$ . If we take  $w_{\mathbf{p}} = 1$ , the AF quasi-particles have a full  $s$  wave gap, while the dSC quasi-particles have a full  $d$  wave gap, with step discontinuity at  $(\pm\pi/2, \pm\pi/2)$  points. For the choice of  $w_{\mathbf{p}} = |\cos p_x - \cos p_y|$ , the dSC quasi-particles have the usual  $\cos p_x - \cos p_y$  gap behavior (Fig.1), while the AF quasi-particles have an anisotropic  $s$  wave gap with nodes at  $(\pm\pi/2, \pm\pi/2)$  points (Fig.2). Because the AF nodes are not “topological”, any small interactions will remove it [11]. In either case, the amplitude of the gaps are the same in both phases.

In addition to the gapped single-particle excitations, the  $SO(5)$  invariant models also have four gapless Goldstone modes. There are ten conserved Noether current density  $L_{\mu}^{ab}(x)$  associated with the  $SO(5)$  symmetry, where  $\mu (= 0, 1, 2)$  is the space-time index. In the small  $q$  limit, exact current conservation leads to the following generalized Ward identity for the vertex function  $\gamma_{\mu}^{ab}(p+q, p)$  ( $p = (\omega, \mathbf{p})$ ) related to this current [10]

$$q^{\mu} \left( \gamma_{\mu}^{ab}(p+q, p) \right)_{\alpha\beta} = -\frac{1}{4} \left( \Gamma_{\alpha\nu}^{ab} G_{\nu\beta}^{-1}(p+q) - G_{\alpha\nu}^{-1}(p) \Gamma_{\nu\beta}^{ab} \right)$$

For a non-zero  $\langle n_{\mathbf{k}}^a \rangle$ , the rhs of the above equation has a finite  $q \rightarrow 0$  limit, therefore, the four vertex functions  $\gamma_{\mu}^{ab}(p+q, p)$  with  $b \neq a$  must have gapless poles in this limit. This result is formulated in the following theorem:

*Theorem 2:* In a  $SO(5)$  invariant model, there are four gapless Goldstone modes associated with spontaneous symmetry breaking.

In the AF phase, there are two spin wave modes and a  $\pi$  doublet ( $4 = 2 + 2$ ). In the dSC phase, there is one SC phase mode, and a  $\pi$  triplet ( $4 = 1 + 3$ ) [1].

As symmetry-breaking perturbations to the above  $SO(5)$  invariant Hamiltonian, we consider two typical terms. One is the coupling to external fields  $B_{ab}$ ,

$$H_{ext} = - \sum_{a < b} B_{ab} L_{ab}. \quad (16)$$

A particular example of this field is the chemical potential  $B_{15} = -2\mu$ , leading to  $H_{\mu} = 2\mu L_{15}$ . The other is the anisotropy energy

$$H_g = - \sum_{\mathbf{p}, \mathbf{p}', \mathbf{q}} \sum_{a=2,3,4} g(\mathbf{q}) (\Psi_{\mathbf{p}}^\dagger \Gamma^a \Psi_{\mathbf{p}+\mathbf{q}}) (\Psi_{\mathbf{p}'}^\dagger \Gamma^a \Psi_{\mathbf{p}'-\mathbf{q}}) \quad (17)$$

between AF and dSC states. To study the spectrum of  $H = H_{\text{kin}} + H_{\text{int},1} + H_{\mu} + H_g$ , we take  $g(\mathbf{q}) = g\delta_{\mathbf{q},\mathbf{Q}}$  and again use mean field approximation. In the dSC phase,  $\langle n_{\mathbf{p}}^a \rangle$  lies in the plane  $(n_1, n_5)$ . We choose it in the  $n_1$  direction. Then the Green’s function is given by

$$G^{SC}(\mathbf{p}, \mathbf{p}', \omega) = \begin{pmatrix} \frac{(\omega + \varepsilon_{\mathbf{p}} - \mu)1\delta_{\mathbf{p},\mathbf{p}'}}{\omega^2 - (\varepsilon_{\mathbf{p}} - \mu)^2 - \Delta_{\mathbf{p}}^2 + i\delta} & \frac{-i\sigma_y \Delta_{\mathbf{p}} \delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2 - (\varepsilon_{\mathbf{p}} + \mu)^2 - \Delta_{\mathbf{p}}^2 + i\delta} \\ \frac{i\sigma_y \Delta_{\mathbf{p}} \delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2 - (\varepsilon_{\mathbf{p}} - \mu)^2 - \Delta_{\mathbf{p}}^2 + i\delta} & \frac{(\omega + \varepsilon_{\mathbf{p}} + \mu)1\delta_{\mathbf{p},\mathbf{p}'}}{\omega^2 - (\varepsilon_{\mathbf{p}} + \mu)^2 - \Delta_{\mathbf{p}}^2 + i\delta} \end{pmatrix}$$

where  $\Delta_{\mathbf{p}} = -16V_1\phi_{\pi}(\mathbf{p})w_{\mathbf{p}}\sum_{\mathbf{k}}w_{\mathbf{k}}\langle n_{\mathbf{k}}^1 \rangle \equiv \Delta_0\phi_{\pi}(\mathbf{p})w_{\mathbf{p}}$  and the  $g$  term drops out because of symmetry mismatch.  $\Delta_0$  is determined by the gap equation  $1 = 16V_1\sum_{\mathbf{k}}\frac{w_{\mathbf{k}}^2}{2E_{\mathbf{k}}}$ , where  $E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} - \mu)^2 + \Delta_{\mathbf{k}}^2}$ . For the choice  $w_{\mathbf{p}} = |\cos p_x - \cos p_y|$ , we have a usual  $d$ -wave gap. In the AF phase,  $\langle n_{\mathbf{p}}^a \rangle$  lies in the  $(n_2, n_3, n_4)$  space. If we pick the  $n_4$  direction, we have

$$G^{AF}(\mathbf{p}, \mathbf{p}', \omega) = \begin{pmatrix} \frac{(\omega + \varepsilon_{\mathbf{p}})1\delta_{\mathbf{p},\mathbf{p}'} + \Delta_{\mathbf{p}}\sigma_z\delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2 - \varepsilon_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2 + i\delta} & 0 \\ 0 & \frac{(\omega - \varepsilon_{\mathbf{p}})1\delta_{\mathbf{p},\mathbf{p}'} + \Delta_{\mathbf{p}}\sigma_z\delta_{\mathbf{p},\mathbf{p}'+\mathbf{Q}}}{\omega^2 - \varepsilon_{\mathbf{p}}^2 - \Delta_{\mathbf{p}}^2 + i\delta} \end{pmatrix}$$

Here  $\omega_{\pm} = \omega \pm \mu$  and  $\Delta_{\mathbf{p}} = -16\sum_{\mathbf{k}}(V_1w_{\mathbf{p}}w_{\mathbf{k}} + g)\langle n_{\mathbf{k}}^4 \rangle \equiv w_{\mathbf{p}}\Delta_0 + \Delta_g$ .  $\Delta_0$  and  $\Delta_g$  are determined by the gap equation  $\Delta_{\mathbf{p}} = 16\sum_{\mathbf{k}}(V_1w_{\mathbf{p}}w_{\mathbf{k}} + g)\frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}$ . The real space form of  $\Delta_{\mathbf{p}}$  has an on-site contribution from  $\Delta_g$  and a long-ranged contribution proportional to  $\Delta_0\phi_M(\mathbf{R})$ . We see that the  $g$  terms leave dSC gap unaffected, while it removes the AF gap node (See Figs.1 and 2). The ground state energy curves are shown in Fig.3. The “superspin flop” transition from AF to dSC occurs at  $\mu = 0$  for  $H_g = 0$ , while it occurs at a finite value of  $\mu_c$  for  $g \neq 0$ . In this case, the AF/dSC transition is first order, with a finite jump in hole density  $x_c$  (See Figs.3 and 4).

While the above pictures are based on the mean field approximation, some exact statements can be made

about the AF/dSC transitions.  $SO(5)$  is a rank 2 algebra, and we can choose  $Q = -L_{15}$  and  $S_z = -L_{23}$  as the members of the Cartan (maximal commutative) subalgebra. In addition, we have the Casimir operator  $C = \sum_{a < b} L_{ab}^2$ , which commutes with all the generators and has eigenvalue  $l(l+3)$ . The set  $(Q, S_z, C)$  forms a Cartesian coordinate system labelling the quantum numbers of all states in the Hilbert space. If we consider only states with even number of electrons, these states form a pyramid, with the  $l = 0$  singlet on top, the  $l = 1$  vector next, and the  $l = 2$  traceless symmetric tensor on the 3rd layer etc. States on the same layer are all connected by repeated actions of the 8 root generators. Therefore,

*Theorem 3:* In  $SO(5)$  invariant models, it is sufficient to diagonalize the Hamiltonian at half-filling with  $Q = 0$  and  $S_z = 0$ . All the other states (with even number of electrons) in the Hilbert space can be obtained from these states through the action of  $(S_x, S_y, \vec{\pi}, \vec{\pi}^\dagger)$ .

In this sense, states at half-filling fully determine the states away from half-filling. In the presence of the  $H_\mu$  term, the  $\pi_\alpha^\dagger$  and  $\pi_\alpha$  operators are exact eigenoperators of the Hamiltonian with eigenvalue  $\pm 2\mu$ . Therefore,  $H_\mu$  commutes with the Casimir operator, and simply shifts the energy of the  $Q \neq 0$  states linearly without changing the wave function of these states. In a system with spontaneous symmetry breaking, lowest states with different  $l$  quantum numbers are separated by inverse system size. In the infinite system limit, these shifts due to chemical potential converge to the parabola as depicted in Fig. 3 [12].

The symmetry-breaking terms,  $H_g$  and  $H_\mu$  produce the mass gap in the Goldstone mode spectrum. For  $H_g = 0$ , the mass of the  $\pi$  triplet mode is exactly  $2|\mu|$ . For finite  $H_g$  we employ the equations of motion (EOM).

As discussed earlier,  $L_{ab}$ 's commute with  $H_{\text{kin}} + H_{\text{int},1}$  for arbitrary  $V_1(\mathbf{p}, \mathbf{p}'; \mathbf{q})$ . The only contributions to the EOM for  $L_{ab}$  come from the  $SO(5)$  symmetry breaking terms,  $H_g$  and  $H_\mu$ :

$$[L_{ab}, H] = -i \sum_c (B_{ac} L_{bc} - B_{bc} L_{ac}) \quad (18)$$

$$- i(\delta_a - \delta_b) \sum_{\mathbf{q}} g(\mathbf{q} + \mathbf{Q}) \{n_a(\mathbf{q}), n_b(-\mathbf{q})\},$$

where  $\delta_a = 1$  for  $a = 2, 3, 4$ , and  $= 0$  otherwise. This is precisely the same equation as that obtained in [1]. EOM for  $n_a$  is given by

$$[n_a, H] = \frac{1}{4} \sum_{\mathbf{p}} (\varepsilon_{\mathbf{p}+\mathbf{Q}} - \varepsilon_{\mathbf{p}}) n_a(\mathbf{p}, 0) + i \sum_b B_{ab} n_b \quad (19)$$

$$- i \sum_{\mathbf{q}, b} (V_1(\mathbf{q} + \mathbf{Q}) + g(\mathbf{q} + \mathbf{Q}) \delta_b) \{n_b(\mathbf{q}), L_{ab}(-\mathbf{q})\}.$$

where  $n_a(\mathbf{q}) = \frac{1}{4} \sum_{\mathbf{p}} n_a(\mathbf{p}, \mathbf{q})$ ,  $L_{ab}(\mathbf{q}) = \frac{1}{8} \sum_{\mathbf{p}} L_{ab}(\mathbf{p}, \mathbf{q})$ , and a form  $V_1(\mathbf{p}, \mathbf{p}'; \mathbf{q}) = -V_1(\mathbf{q})$  and the property  $V_1(\mathbf{Q} - \mathbf{q}) = V_1(\mathbf{Q} + \mathbf{q})$  has been assumed.

Let us now consider the case where only the chemical potential  $B_{15} = -2\mu$  is introduced as external fields, and consider only the collective part in equation (19) [13,14]. In the dSC state, we take  $\langle n_a(\mathbf{q}) \rangle = \langle n_1 \rangle \delta_{a,1} \delta_{\mathbf{q},0}$ , and linearize eq.(19) to obtain

$$\dot{n}_a = 2V_1 \langle n_1 \rangle L_{1a}, \quad (a = 2, 3, 4) \quad (20)$$

$$\dot{n}_5 - 2\mu = 2V_1 \langle n_1 \rangle L_{15}, \quad (21)$$

where  $V_1 \equiv V_1(\mathbf{Q})$ . These equations should be compared with  $\chi \dot{n}_a = L_{1a}$  with  $a = 2, 3, 4$  and  $\chi(\dot{n}_5 - 2\mu) = L_{15}$  derived from the nonlinear  $\sigma$ -model [1]. Equations (18), (21), and (21) can be combined to give  $\dot{n}_5 = 0$  and  $\ddot{n}_a = 4(\mu_c^2 - \mu^2)n_a$ , where  $\mu_c = \langle n_1 \rangle \sqrt{gV_1}$  and  $g \equiv g(\mathbf{Q})$ . Therefore, the energy of the triplet  $\pi$  excitations in the dSC state is given by  $\omega_0 = 2\sqrt{\mu_c^2 - \mu^2}$ , which is also consistent with the result of [1]. Similar calculation in the AF state gives the energies of the  $\pi$  doublet  $\omega_0 = 2\langle n_4 \rangle \sqrt{g(V_1 + g)} \pm 2\mu$ , where we assumed AF ordering along  $n_4$ . We therefore see that the two symmetry breaking terms  $g$  and  $\mu$  partially compensate each other for the  $\pi$  triplet and  $Q = -2$   $\pi$  doublet.

In conclusion we have constructed a class of electron models with exact  $SO(5)$  symmetry. Both the fermionic single particle and bosonic collective modes of the AF and dSC phases are in one-to-one correspondence to each other. Energy levels are classified according to the  $SO(5)$  quantum numbers and the level crossing at the AF/dSC transition can be followed in detail. The fermionic single-particle gaps do not close at the AF/dSC transition, but instead rotate from one direction in superspin space to another. It is amusing to ask what experimental results this kind of ideal models would predict. The phase diagram would be identical to the one depicted in Fig. 1A of reference [1], with an insulating AF gap and a finite jump in chemical potential at half filling, phase separation or stripe ordering for doping range  $0 < x < x_c$ , a low energy spin triplet  $\pi$  resonance in the dSC phase, and a “pseudogap” behavior in the high temperature phase above the bi-critical point, where the gap direction fluctuates (rotates) between AF and dSC character.

The authors are deeply indebted to Prof. C. Henley for generous sharing of his ideas. We would like to thank Prof. R.B. Laughlin for focusing our attention on the study of the AF/dSC transition problem, Prof. I. Affleck, E. Fradkin, D. Gross, M. Oshikawa, J. Preskill, A. Zee for useful discussions on  $SO(5)$  Lie algebra and Prof. J. Berlinsky, H. Fukuyama, C. Kallin, S. Kivelson and D. Scalapino for general discussion on the high-Tc problem. This work is supported by the NSF under grant numbers DMR-9400372 and DMR-9522915. S.R. acknowledges the support from CNPq (Brazilian Research Council).

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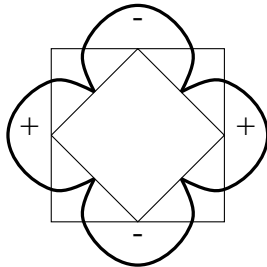


FIG. 1. The superconducting (dSC) gap.

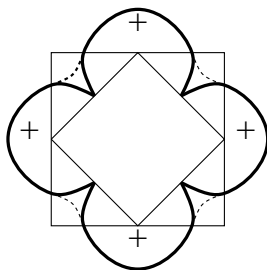


FIG. 2. The antiferromagnetic (AF) gap. The solid (dotted) line is for the case with (without)  $SO(5)$  symmetry.

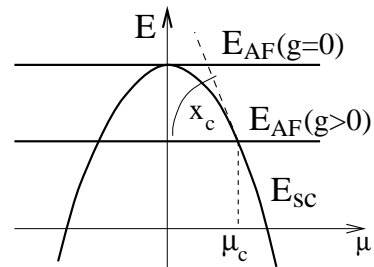


FIG. 3. The groundstate energy  $E_G$  in both AF and SC phases as functions of  $\mu$ .

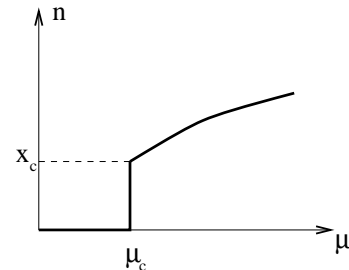


FIG. 4. The electron density  $x$  versus  $\mu$  in the presence of anisotropy energy  $H_g$ .